Abelianisation and differential structures

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joint with JS Lemay and Tim Van der Linden

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Cartesian Differential Categories : motivation

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$$\frac{f \in \mathrm{Smooth}(\mathbb{R}^n, \mathbb{R}^m)}{\dfrac{(f_1, \ldots, f_m) \in \mathrm{Smooth}(\mathbb{R}^n, \mathbb{R})^{\times m}}{}}$$

$$Df = \begin{pmatrix} \frac{\partial f_1}{x_1} & \ldots & \frac{\partial f_1}{x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{x_1} & \ldots & \frac{\partial f_m}{x_n} \end{pmatrix}$$

Cartesian Differential Categories : motivation

$$\frac{\dfrac{f \in \mathrm{Smooth}(\mathbb{R}^n, \mathbb{R}^m)}{(f_1, \ldots, f_m) \in \mathrm{Smooth}(\mathbb{R}^n, \mathbb{R})^{\times m}}}{}$$

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$$\begin{aligned} Df : \quad & \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m \\ (\vec{x}, \vec{y}) \quad & \mapsto Df(\vec{x}, \vec{y}) := D_{\vec{x}} f(\vec{y}) = \begin{pmatrix} \frac{\partial f_1}{x_1}(\vec{x}) & \ldots & \frac{\partial f_1}{x_n}(\vec{x}) \\ \vdots & & \vdots \\ \frac{\partial f_m}{x_1}(\vec{x}) & \ldots & \frac{\partial f_m}{x_n}(\vec{x}) \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \end{aligned}$$

Cartesian Differential Categories : preliminaries

Definition (Blute, Cockett, Seely, '09)

A **left additive** category is a category $\mathbb{X}$ where for all objects A, B, $\mathbb{X}(A, B)$ is a commutative monoid and :

$$(f+g)\circ h = f\circ h + g\circ h, \qquad 0\circ f = 0.$$

A morphism f in $\mathbb{X}$ is **linear** if, moreover :

$$f\circ(g+h) = f\circ g + f\circ h, \qquad f\circ 0 = 0.$$

Definition (Blute, Cockett, Seely, '09)

A **Cartesian left additive category** is a left additive category equipped with all finite product and where projection maps are linear.

Cartesian Differential Categories : definition

Definition (Blute, Cockett, Seely, '09)

A **Cartesian Differential Category** is a Cartesian left additive category $\mathbb{X}$ equipped with a **differential combinator** of the form :

$$\frac{f : A \to B}{Df : A \times A \to B}$$

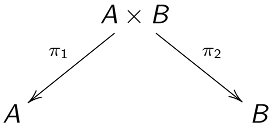
Satisfying [CD.1] to [CD.7].

Ex : Smooth.

[CD.1] : D is linear in f,

[CD.2] : Df is linear in $\vec{y}$,

[CD.3] : $D(\mathsf{Id}_X)(\vec{x}, -) = \mathsf{Id}_X$, $D(\pi_i)(\vec{x}, -) = \pi_i$



$$\frac{B_1 \xleftarrow{f} A \xrightarrow{g} B_2}{\langle f, g\rangle : A \to B_1 \times B_2}$$

[CD.4] : $D(\langle f, g\rangle) = \langle Df, Dg\rangle$

[CD.5] : $D(g \circ f)(\vec{x}, \vec{y}) = Dg(f(\vec{x}), Df(\vec{x}, \vec{y}))$ (Chain Rule !)

[CD.6] : $D(Df)(\vec{x}, 0, 0, \vec{y}) = Df(\vec{x}, \vec{y})$

[CD.7] : $D(Df)(\vec{x}, \vec{y}, \vec{z}, \vec{t}) = D(Df)(\vec{x}, \vec{z}, \vec{y}, \vec{t})$
(Mixed Partial derivatives commute)

CDCs to Tangent categories

Proposition (Cockett, Cruttwell, '14)

If $\mathbb{X}$ is a Cartesian differential category, then there is a tangent structure on $\mathbb{X}$ such that :

$$T(X) = X \times X, \qquad T(f)(x, y) = (f(x), Df(x, y)),$$

$$p_X(x, y) = x, \qquad s_X(x, y, z) = (x, y + z), \qquad z_X(x) = (x, 0),$$

$$l_X(x, y) = (x, 0, 0, y), \qquad c_X(x, y, z, t) = (x, z, y, t)$$

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Definition (Cockett, Cruttwell, '18)

A **differential object** in a tangent category $\mathbb{X}$ is an object X such that :

$$TX \cong X \times X$$

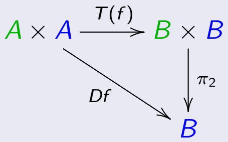
Differential objects in $\mathbb{X}$ form a category $\mathrm{Diff}_{\mathbb{X}}$.

Tangent categories to CDCs

Proposition (Cockett, Cruttwell, '18)

$\mathrm{Diff}_{\mathbb{X}}$ comes naturally equipped with a CDC structure where :

$$f : A \to B$$



A tangent structure for groups :

Theorem (I., Lemay, V.d.Linden)

There is a tangent structure on Grp *such that :*

$$T : \mathrm{Grp} \to \mathrm{Grp} \qquad G \mapsto G \times \mathrm{Ab}(G) \qquad f \mapsto (f, \mathrm{Ab}(f)),$$

$$p_G : T(G) \to G \qquad (g, [h]) \mapsto g$$

$$s_G : T_2(G) \to T(G) \qquad (g, [h_1], [h_2]) \mapsto (g, [h_1] + [h_2]),$$

$$z_G : G \to T(G) \qquad g \mapsto (g, 0),$$

$$l_G : T(G) \to T^2(G) \qquad (g, [h]) \mapsto (g, 0, 0, [h]),$$

$$c_G : T^2(G) \to T^2(G) \qquad (g, [h_1], [h_2], [h_3]) \mapsto (g, [h_2], [h_1], [h_3]).$$

Grp is almost a CDC

$$T(G) = G \times \text{something}$$

Generalised Cartesian Differential Categories (GCDC)

Definition (Cruttwell '18)

A *GCDC* is the data of :

- A cartesian category $\mathbb{X}$,
- For each object X in $\mathbb{X}$, a commutative monoid object $L(X)$, such that :

$$LL(X) = L(X) \qquad L(X \times Y) = L(X) \times L(Y),$$

- For each morphism $f : X \to Y$, a *directional derivative* :

$$D[f] : X \times L(X) \to L(Y),$$

satisfying [GCD.1] to [GCD.7].

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satisfying [GCD.1] to [GCD.7].

Remarks :

- No "left-additiveness",
- The linearisation L is not necessarily functorial.
- Equalities can be relaxed.

GCDCs vs Tangent categories

Proposition (Cruttwell, '18)

If $\mathbb{X}$ is a GCDC, then there is a tangent structure on $\mathbb{X}$ such that :

$$T(X) = X \times L(X), \qquad T(f)(x,y) = (f(x), Df(x,y)),$$

$$p_X(x,y) = x, \qquad s_X(x,y,z) = (x, y+z), \qquad z_X(x) = (x,0),$$

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Definition

A **parallelisable object** in a tangent category $\mathbb{X}$ is an object X such that :

$$TX \cong X \times \text{something}$$

Parallelisable objects in $\mathbb{X}$ form a category $\mathrm{Parall}_{\mathbb{X}}$.

Theorem (JS promised me this !)

$\mathrm{Parall}_{\mathbb{X}}$ *has a GCDC structure.*

A GCDC structure for groups

Consider :

$$L(G) := \mathrm{Ab}(G)$$

$$\frac{f : G \to H}{\begin{array}{r} D[f] : G \times \mathrm{Ab}(G) \to \mathrm{Ab}(H) \\ (g_1, [g_2]) \mapsto [f(g_2)] \end{array}}$$

Theorem (I., Lemay, V.d.Linden)

(Grp, L, D) *is a GCDC.*

A GCDC structure for groups

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Theorem (I., Lemay, V.d.Linden)

(Grp, L, D) *is a GCDC.*

Is "abelianisation" an example of linearisation in general ?

What is an "abelianisation" functor ?

What did we use for groups ?

$$\mathrm{Ab} \underset{\hookrightarrow}{\overset{\mathsf{Ab}}{\underset{\perp}{\longleftarrow}}} \mathrm{Grp}, \quad \text{where}$$

- Grp has finite products
- ~~Ab is abelian~~ Ab is semi-linear
- Ab is (strongly) idempotent and preserves products (strongly).

Reflective subcategories

Definition ('60s)

A **reflective** subcategory of a category $\mathbb{X}$ is a full subcategory $\mathbb{Y} \hookrightarrow \mathbb{X}$ with a left adjoint $\mathbb{Y} \xleftarrow{L} \mathbb{X}$ called the **reflector**.

A **linear reflective** subcategory of $\mathbb{X}$ is a reflective subcategory

$$\mathbb{L} \underset{\hookrightarrow}{\overset{L}{\underset{\perp}{\leftarrow}}} \mathbb{X}$$

where $\mathbb{L}$ is semi-additive.

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where $\mathbb{L}$ is semi-additive.

Remark : any object X in $\mathbb{L}$ is a commutative monoid for the "fold" map $X \oplus X \to X$ and the unit $(1_X, 0) : X \to X \oplus X$

Exercise : L is strongly idempotent (Hint : the counit is a natural iso)

Unital categories

Definition (Bourne '96)

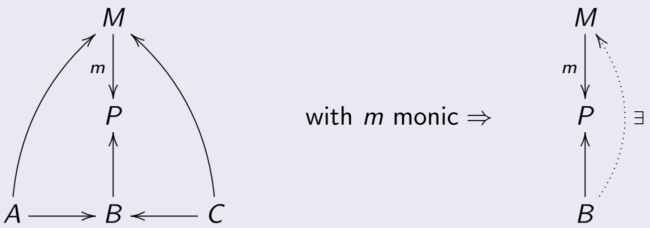
A category $\mathbb{X}$ is *unital* if it admits finite products and the maps :

$$X \xrightarrow{(1_X,0)} X \times Y \xleftarrow{(0,1_Y)} Y$$

are **jointly strongly epic**.

Definition (Kelly '60s ?)

A cospan $A \longrightarrow B \longleftarrow C$ is **strongly jointly epic** if :



Linear reflective subcat. and Unital cat.

Lemma

Let $\mathbb{X}$ be a unital category and $\mathbb{L} \underset{\perp}{\overset{L}{\leftrightarrows}} \mathbb{X}$ be a linear reflective subcategory. Then, L preserves products strongly.

Idea : $L(X) \xrightarrow{L(1_X,0)} L(X \times Y) \xleftarrow{L(0,1_Y)} L(Y)$ is a coproduct $L(X) + L(Y)$.

~~Abelianisation~~ Linearisation gives a GCDC

Theorem (I., Lemay, V.d.Linden)

Let $\mathbb{X}$ be a unital category and $\mathbb{L} \underset{\perp}{\overset{L}{\rightleftarrows}} \mathbb{X}$ *be a linear reflective subcategory. Then,* $(\mathbb{X}, L, D)$ *is a GCDC, with* $D[f] = L(f) \circ \pi_1$.

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Let $\mathbb{X}$ *be a unital category and* $\mathbb{L} \underset{\perp}{\overset{L}{\leftrightarrows}} \mathbb{X}$ *be a linear reflective subcategory. Then,* $(\mathbb{X}, L, D)$ *is a GCDC, with* $D[f] = L(f) \circ \pi_1$*.*

Examples : any *semi-abelian* category has a GCDC structure.

- Grp,
- non-unital rings, with : $L(R) = R/R^2$,
- Lie algebras, with $L(\mathfrak{g}) = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$,
- cocomutative Hopf algebras (in char. 0) with $L(H) = H/\langle \mu(a \otimes b) = \mu(b \otimes a) \rangle$

Future work : going the other way

Is there a tangent category $(\mathbb{X}, T)$ such that :

- Not all objects are parallelisable,
- Parallelisable objects $= \mathrm{Grp}$,
- Differentiable objects $=$ Ab,
- . . . Is interesting ?

Tentative : Completely Non-Abelian Rings (CNARs)
$\rightarrow$ Rings where $+$ is not commutative. . .
$\rightarrow$ Groups G with internal cross pairings $G \otimes G \rightarrow G$?
Associative, or Lie-like ?

Does not work well *so far*.

Thank you !